

Golden-Thompson's inequality for deformed exponentials

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Abstract

Deformed logarithms and their inverse functions, the deformed exponentials, are important tools in the theory of non-additive entropies and non-extensive statistical mechanics. We formulate and prove counterparts of Golden-Thompson's trace inequality for q -exponentials with parameter q in the interval $[1, 3]$.

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1 Introduction and main result

Tsallis [7] generalised in 1988 the standard Boltzmann-Gibbs entropy to a non-extensive quantity S_q depending on a parameter q . In the quantum version it is given by

$$S_q(\rho) = \frac{1 - \text{Tr } \rho^q}{q - 1} \quad q \neq 1,$$

where ρ is a density matrix. It has the property that $S_q(\rho) \rightarrow S(\rho)$ for $q \rightarrow 1$, where $S(\rho) = -\text{Tr } \rho \log \rho$ is the von Neumann entropy. The Tsallis entropy may be written on a similar form

$$S_q(\rho) = -\text{Tr } \rho \log_q(\rho),$$

where the deformed logarithm \log_q is given by

$$\log_q x = \int_1^x t^{q-2} dt = \begin{cases} \frac{x^{q-1} - 1}{q-1} & q > 1 \\ \log x & q = 1 \end{cases}$$

for $x > 0$. The deformed logarithm is also denoted the q -logarithm. The inverse function \exp_q is called the q -exponential and is given by

$$\exp_q(x) = (x(q-1) + 1)^{1/(q-1)} \quad \text{for } x > \frac{-1}{q-1}.$$

The q -logarithm and the q -exponential functions converge, respectively, to the logarithmic and the exponential functions for $q \rightarrow 1$.

The aim of this article is to generalise Golden-Thompson's trace inequality [2, 6] to deformed exponentials. The main result is the following:

Theorem 1.1. *Let A and B be positive definite matrices.*

(i) *If $1 \leq q < 2$ then*

$$\text{Tr } \exp_q(A + B) \leq \text{Tr } \exp_q(A)^{2-q} (A(q-1) + \exp_q B).$$

(ii) *If $2 \leq q \leq 3$ then*

$$\text{Tr } \exp_q(A + B) \geq \text{Tr } \exp_q(A)^{2-q} (A(q-1) + \exp_q B).$$

Notice that we for $q = 1$ recovers Golden-Thomson's trace inequality

$$\text{Tr } \exp(A + B) \leq \text{Tr } \exp(A) \exp(B).$$

This inequality is valid for arbitrary self-adjoint matrices A and B . However, it is sufficient to know the inequality for positive definite matrices, since the general form follows by multiplication with positive numbers.

2 Preliminaries

We collect a few well-known results that we are going to use in the proof of the main theorem.

The q -logarithm is a bijection of the positive half-line onto the open interval $-(q-1)^{-1}, \infty$, and the q -exponential is consequently a bijection of the interval $-(q-1)^{-1}, \infty$ onto the positive half-line. For $q > 1$ we may thus safely apply both the q -logarithm and the q -exponential to positive definite operators. We also notice that

$$(1) \quad \frac{d}{dx} \log_q(x) = x^{q-2} \quad \text{and} \quad \frac{d}{dx} \exp_q(x) = \exp_q(x)^{2-q}.$$

The proof of the following lemma is rather easy and may be found in [4, Lemma 5].

Lemma 2.1. *Let $\varphi: \mathcal{D} \rightarrow \mathcal{A}_{sa}$ be a map defined in a convex cone \mathcal{D} in a Banach space X with values in the self-adjoint part of a C^* -algebra \mathcal{A} . If φ is Fréchet differentiable, convex and positively homogeneous then*

$$d\varphi(x)h \leq \varphi(h).$$

for $x, h \in \mathcal{D}$.

Let H be any $n \times n$ matrix. The map

$$A \rightarrow \text{Tr}(H^* A^p H)^{1/p},$$

defined in positive definite $n \times n$ matrices, is concave for $0 < p \leq 1$ and convex for $1 \leq p \leq 2$, cf. [1, Theorem 1.1]. By a slight modification of the construction given in Remark 3.2 in the same reference, cf. also [3], we obtain that the mapping

$$(2) \quad (A_1, \dots, A_k) \rightarrow \text{Tr}(H_1^* A_1^p H_1 + \dots + H_k^* A_k H_k)^{1/p},$$

defined in k -tuples of positive definite $n \times n$ matrices, is concave for $0 < p \leq 1$ and convex for $1 \leq p \leq 2$; for arbitrary $n \times n$ matrices H_1, \dots, H_k .

3 Deformed trace functions

Theorem 3.1. *Let H_1, \dots, H_k be matrices with $H_1^* H_1 + \dots + H_k^* H_k = 1$ and define the function*

$$(3) \quad \varphi(A_1, \dots, A_k) = \text{Tr} \exp_q \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i \right)$$

in k -tuples of positive definite matrices. Then φ is positively homogeneous of degree one. It is concave for $1 \leq q \leq 2$ and convex for $2 \leq q \leq 3$.

Proof. For $q > 1$ we obtain

$$\begin{aligned}
\varphi(A_1, \dots, A_k) &= \text{Tr} \exp_q \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i \right) \\
&= \text{Tr} \left((q-1) \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i \right) + 1 \right)^{1/(q-1)} \\
&= \text{Tr} \left((q-1) \left(\sum_{i=1}^k H_i^* \frac{A_i^{q-1} - 1}{q-1} H_i \right) + 1 \right)^{1/(q-1)} \\
&= \text{Tr} \left(\sum_{i=1}^k H_i^* (A_i^{q-1} - 1) H_i + 1 \right)^{1/(q-1)} \\
&= \text{Tr} (H_1^* A_1^{q-1} H_1 + \dots + H_k^* A_k^{q-1} H_k)^{1/(q-1)}.
\end{aligned}$$

From this identity it follows that φ is positively homogeneous of degree one. The concavity for $1 < q \leq 2$ and the convexity for $2 \leq q \leq 3$ now follows from (2). The statement for $q = 1$ follows by letting q tend to one. **QED**

Corollary 3.2. *Let L be positive definite, and let H_1, \dots, H_k be matrices such that $H_1^* H_1 + \dots + H_k^* H_k \leq 1$. Then the function*

$$\varphi(A_1, \dots, A_k) = \text{Tr} \exp_q (L + H_1^* \log_q(A_1) H_1 + \dots + H_k^* \log_q(A_k) H_k),$$

defined in k -tuples of positive definite matrices, is concave for $1 \leq q \leq 2$ and convex for $2 \leq q \leq 3$.

Proof. We may without loss of generality assume $H_1^* H_1 + \dots + H_k^* H_k < 1$ and put $H_{k+1} = (1 - (H_1^* H_1 + \dots + H_k^* H_k))^{1/2}$. We then have

$$H_1^* H_1 + \dots + H_k^* H_k + H_{k+1}^* H_{k+1} = 1$$

and may use the preceding theorem to conclude that the function

$$(A_1, \dots, A_{k+1}) \rightarrow \text{Tr} \exp_q (H_1^* \log_q(A_1) H_1 + \dots + H_{k+1}^* \log_q(A_{k+1}) H_{k+1})$$

of $k+1$ variables is concave for $1 \leq q \leq 2$ and convex for $2 \leq q \leq 3$. Since H_{k+1} is invertible we may choose

$$A_{k+1} = \exp_q (H_{k+1}^{-1} L H_{k+1}^{-1})$$

which makes sense since $H_{k+1}^{-1} L H_{k+1}^{-1}$ is positive definite. Concavity for $1 \leq q \leq 2$ and convexity for $2 \leq q \leq 3$ in the first k variables of the above function then yields the result. **QED**

Setting $q = 1$ we recover in particular [5, Theorem 3].

Corollary 3.3. *Let H_1, \dots, H_k be matrices with $H_1^* H_1 + \dots + H_k^* H_k \leq 1$, and let L be self-adjoint. The trace function*

$$(A_1, \dots, A_k) \rightarrow \text{Tr} \exp(L + H_1^* \log(A_1) H_1 + \dots + H_k^* \log(A_k) H_k)$$

is concave in positive definite matrices.

Corollary 3.4. *The trace function φ defined in (3) satisfies*

$$\varphi(B_1, \dots, B_k) \leq \text{Tr} \exp_q \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i \right)^{2-q} \sum_{j=1}^k H_j^* (d \log_q(A_j) B_j) H_j$$

for $1 \leq q \leq 2$ and

$$\varphi(B_1, \dots, B_k) \geq \text{Tr} \exp_q \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i \right)^{2-q} \sum_{j=1}^k H_j^* (d \log_q(A_j) B_j) H_j$$

for $2 \leq q \leq 3$, where A_1, \dots, A_k and B_1, \dots, B_k are positive definite matrices.

Proof. For $1 \leq q \leq 2$ we obtain

$$d\varphi(A_1, \dots, A_k)(B_1, \dots, B_k) \geq \varphi(B_1, \dots, B_k)$$

by Lemma 2.1. By the chain rule for Fréchet differentiable mappings between Banach spaces we therefore obtain

$$\begin{aligned} \varphi(B_1, \dots, B_k) &\leq \sum_{j=1}^k d_j \varphi(A_1, \dots, A_k) B_j \\ &= \sum_{j=1}^k \text{Tr} \, d \exp_q \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i \right) H_j^* (d \log_q(A_j) B_j) H_j \\ &= \sum_{j=1}^k \text{Tr} \exp_q \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i \right)^{2-q} H_j^* (d \log_q(A_j) B_j) H_j \end{aligned}$$

where we used the identity $\text{Tr} \, df(A)B = \text{Tr} \, f'(A)B$ valid for differentiable functions. This proves the first assertion. The result for $2 \leq q \leq 3$ follows similarly. **QED**

4 Proof of the main theorem

In order to prove Theorem 1.1 (i) we set $k = 2$ in Corollary 3.4 and obtain

$$\varphi(B_1, B_2) \leq \text{Tr} \exp_q(X)^{2-q} (H_1^*(d\log_q(A_1)B_1)H_1 + H_2^*(d\log_q(A_2)B_2)H_2)$$

for $1 \leq q \leq 2$ and positive definite matrices A_1, A_2 and B_1, B_2 where

$$X = H_1^* \log_q(A_1)H_1 + H_2^* \log_q(A_2)H_2.$$

If we set $A_1 = B_1$ and $A_2 = 1$ the inequality reduces to

$$\varphi(B_1, B_2) \leq \text{Tr} \exp_q(H_1^* \log_q(B_1)H_1)^{2-q} (H_1^* B_1^{q-1} H_1 + H_2^* B_2 H_2).$$

We now set $H_1 = \varepsilon^{1/2}$ for $0 < \varepsilon < 1$, and to fixed positive definite matrices L_1 and L_2 we choose B_1 and B_2 such that

$$\begin{aligned} L_1 &= H_1^* \log_q(B_1)H_1 = \varepsilon \log_q(B_1) \\ L_2 &= H_2^* \log_q(B_2)H_2 = (1 - \varepsilon) \log_q(B_2). \end{aligned}$$

It follows that

$$B_1 = \exp_q(\varepsilon^{-1}L_1) \quad \text{and} \quad B_2 = \exp_q((1 - \varepsilon)^{-1}L_2).$$

Inserting in the inequality we now obtain

$$\begin{aligned} &\text{Tr} \exp_q(L_1 + L_2) \\ &\leq \text{Tr} \exp_q(L_1)^{2-q} (\varepsilon \exp_q(\varepsilon^{-1}L_1)^{q-1} + (1 - \varepsilon) \exp_q((1 - \varepsilon)^{-1}L_2)) \\ &= \text{Tr} \exp_q(L_1)^{2-q} (L_1(q - 1) + \varepsilon + (1 - \varepsilon) \exp_q((1 - \varepsilon)^{-1}L_2)). \end{aligned}$$

This expression decouple L_1 and L_2 and reduces the minimisation problem over ε to the commutative case. We furthermore realise that minimum is obtained by letting ε tend to zero and that

$$\lim_{\varepsilon \rightarrow 0} (1 - \varepsilon) \exp_q((1 - \varepsilon)^{-1}L_2) = \exp_q(L_2).$$

We finally replace L_1 and L_2 with A and B . This proves the first statement in Theorem 1.1.

The proof of the second statement is virtually identical to the proof of the

first. Since now $2 \leq q \leq 3$ the second inequality in Corollary 3.4 applies. Setting $k = 2$ and applying the same substitutions as in the proof of the first statement we arrive at the inequality

$$\begin{aligned} & \text{Tr} \exp_q(L_1 + L_2) \\ & \geq \text{Tr} \exp_q(L_1)^{2-q} (L_1(q-1) + \varepsilon + (1-\varepsilon) \exp_q((1-\varepsilon)^{-1}L_2)). \end{aligned}$$

Since $2 \leq q \leq 3$ the function

$$\varepsilon \rightarrow \varepsilon + (1-\varepsilon) \exp_q((1-\varepsilon)^{-1}L_2)$$

is now decreasing, and we thus maximise the right hand side in the above inequality by letting ε tend to zero. This proves the second statement in Theorem 1.1.

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